Limited and Illimited Linear Difference Equations of the Second Order with Periodic Coefficients.

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In this paper, I show how a method developed by A. Liapounoff * for the linear differential equation of the second order can be extended to the difference equation in which the independent variable is restricted to integral values. Certain portions of Liapounoff's work can be applied to the difference equation, with changes which are in no wise fundamental. I shall consequently state some results without proof, the proofs being readily supplied from the paper of Liapounoff. The fundamental theorem of Liapounoff, the proof of which, as given by him, is exceedingly difficult and long, covering some sixteen pages, when stated for the difference equation admits a proof both short and simple. This simplicity does not, however, extend throughout the theory, as the formulas to be used in the calculations are usually more difficult to obtain and somewhat more complicated for the difference than for the differential equation

§ 1. *A Necessary and Sufficient Condition that the Equation be Limited.*

Given

$$y(i+2) + M(i)\,y(i+1) + y(i) = 0, \tag{1}$$

where the function $M(i)$ is real and defined for all integral values of the argument, and satisfies the relation $M(i+\omega) \equiv M(i)$.

Let y_1 and y_2 be two linearly independent solutions. Then, as $y_1(i+\omega)$ and $y_2(i+\omega)$ are also solutions,

$$\left.\begin{aligned} y_1(i+\omega) &\equiv a_{11}\,y_1(i) + a_{12}\,y_2(i), \\ y_2(i+\omega) &\equiv a_{21}\,y_1(i) + a_{22}\,y_2(i), \end{aligned}\right\} \tag{2}$$

where a_{11}, a_{12}, a_{21} and a_{22} are constants; and since the determinant

$$\begin{vmatrix} y_1(i) & y_1(i+1) \\ y_2(i) & y_2(i+1) \end{vmatrix}$$

is a constant,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 1.$$

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* *Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg,* 8e Série, Tome 13, 1902.

Consequently, the characteristic equation* of (1),

$$\begin{vmatrix} a_{11}-\rho & a_{12} \\ a_{21} & a_{22}-\rho \end{vmatrix} = 0,$$

reduces to $\rho^2 - (a_{11} + a_{22})\,\rho + 1 = 0$, which we write

$$\rho^2 - 2A\rho + 1 = 0. \tag{3}$$

From the first of equations (2),

$$y_1(i+\omega) = 2Ay_1(i) + a_{12}y_2(i) - a_{22}y_1(i). \tag{4}$$

Write equations (2) in the form

$$\left.\begin{aligned} y_1(i) &= a_{11}y_1(i-\omega) + a_{12}y_2(i-\omega), \\ y_2(i) &= a_{21}y_1(i-\omega) + a_{22}y_2(i-\omega). \end{aligned}\right\} \tag{5}$$

Solve (5) for $y_1(i-\omega)$ and substitute in (4). We get

$$y_1(i+\omega) + y_1(i-\omega) = 2Ay_1(i).$$

But A, being a coefficient of the characteristic equation, is independent of the particular fundamental system of solutions chosen. Hence, when y is any solution of (1),

$$y(i+\omega) + y(i-\omega) \equiv 2Ay(i). \tag{6}$$

We call A the characteristic constant of the difference equation, (1).

From (6), proceeding exactly as is done by Liapounoff in the case of the differential equation, we obtain the following theorem:

THEOREM I. *If $A^2 > 1$, all solutions of* (1), *not identically zero, are illimited.*† *If $A^2 < 1$, all solutions of* (1) *are limited. If $A = 1$, there exists at least one solution, not identically zero, having the period ω; all solutions not having the period ω are illimited. If $A = -1$, there exists at least one solution, not identically zero, satisfying the relation $y(i+\omega) \equiv -y(i)$; all solutions not having the period 2ω are illimited.*

§ 2. *The Finite Series*, $1 - A_1 + A_2 - \ldots + (-1)^\omega A_\omega$.

The problem of the calculation of A next presents itself.

Let $f(i)$ and $\phi(i)$ be the two solutions of (1) such that

$$f(0) = 1, \quad \Delta f(0) = 0;$$
$$\phi(0) = 0, \quad \Delta\phi(0) = 1.$$

From (6), $f(\omega) + f(-\omega) = 2A$. Moreover, $\Delta\phi(\omega)f(i+\omega) - \Delta f(\omega)\phi(i+\omega)$ is a linear combination of $f(i+\omega)$ and $\phi(i+\omega)$, two solutions of (1), and hence

* The characteristic equation of (1) is the analogue of the characteristic equation of the differential equation, $\frac{d^2y}{dx^2} + m(x)\,y = 0$, where $m(x)$ is periodic. Compare Floquet, *Ann. Sci. de l'École Normale Supérieure*, 2e Série, XII, p. 47.

† A solution of (1) is said to be limited if it remains finite as i becomes infinite, and to be illimited in the contrary case.

is itself a solution. Moreover, it and its first difference at 0 are equal respectively to $f(0)$ and $\Delta f(0)$; hence

$$f(i) \equiv \Delta\phi(\omega) f(i+\omega) - \Delta f(\omega)\phi(i+\omega). \tag{7}$$

From (7), $f(-\omega) = \Delta\phi(\omega)$, and hence

$$2A = f(\omega) + \Delta\phi(\omega). \tag{8}$$

If ω is small, the calculation of A from (8) is easy. We calculate $f(2)$, $f(3), \ldots, f(\omega)$ successively from (1), then $\phi(2), \phi(3), \ldots, \phi(\omega+1)$, and substitute in (8).

If ω is large, this process is tedious, and for very large values of ω is prohibitive. In the following pages a process, analogous to that employed by Liapounoff for the differential equation, is developed for the treatment of this case. We begin by writing the difference equation in the form

$$\Delta^2 y(i) + p(i)\, y(i+1) = 0, \tag{9}$$

where $p(i)$ replaces $M(i) + 2$.

Treat $f(i)$ and $\phi(i)$ by a method of successive approximations* similar to that employed in existence theorems for differential equations. Denote the successive terms in the two series by $f_0, -f_1, \ldots, (-1)^n f_n, \ldots$, and $\phi_0, -\phi_1, \ldots, (-1)^n \phi_n$ respectively. Adopting the convention $\sum_{i=k}^{g} F(i) = 0$, $k > g$; when $i \geqq 0$,

$$\phi_0(i) = i,$$

$$\phi_1(i) = \sum_{i_1=0}^{i-1} \sum_{i_2=0}^{i_1-1} p(i_2)(i_2+1),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\phi_n(i) = \sum_{i_1=0}^{i-1} \sum_{i_2=0}^{i_1-1} \sum_{i_3=0}^{i_2} \cdots \sum_{i_{2n-1}=0}^{i_{2n-2}} \sum_{i_{2n}=0}^{i_{2n-1}-1} p(i_2)\, p(i_4) \cdots p(i_{2n})(i_{2n}+1), \tag{10}$$

$$f_0(i) = 1,$$

$$\ldots\ldots\ldots,$$

$$f_n(i) = \sum_{i_1=0}^{i-1} \sum_{i_2=0}^{i_1-1} \sum_{i_3=0}^{i_2} \cdots \sum_{i_{2n-1}=0}^{i_{2n-2}} \sum_{i_{2n}=0}^{i_{2n-1}-1} p(i_2)\, p(i_4) \cdots p(i_{2n}). \tag{11}$$

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* Consider equation (9). Form successive approximations for a solution, y, such that $y(a) = c_0$, $y(a+1) = c_1$, where a is any integer, and c_0 and c_1 arbitrary constants. Assuming y_{n-1} as known, we determine y_n from $\Delta^2 y_n(i) = -p(i)\, y_{n-1}(i+1)$, subject to the conditions $y_n(a) = c_0$, $y_n(a+1) = c_1$. Choose $y_0 \equiv (i-a)(c_1 - c_0) + c_0$, and let $z_n(i) = y_n(i) - y_{n-1}(i)$ when $n \geqq 1$, $z_0(i) \equiv y_0(i)$. Clearly, when $n \geqq 1$,

$$\Delta^2 z_n(i) = -p(i)\, z_{n-1}(i+1) \text{ and } z_n(a) = z_n(a+1) = 0. \tag{j}$$

Adopting the convention $\sum_{i=k}^{g} F(i) = 0$, $k > g$, we have, when $n \geqq 1$ and $i \leqq a$,

$$z_n(i) = -\sum_{i=a}^{i-1} \sum_{i=a}^{i-1} p(i)\, z_{n-1}(i+1). \tag{jj}$$

From (jj) it is immediate that if $z_{n-1}(\bar{i}) = 0$ when $\bar{i} = a, a+1, \ldots, i-1$, $z_n(\bar{i}) = 0$ when $\bar{i} = a$, $a+1, \ldots, i$. But $z_1(a) = z_1(a+1) = 0$. Hence $z_2(a) = z_2(a+1) = z_2(a+2) = 0$, and in general $z_n(a) = z_n(a+1) = \ldots = z_n(a+n) = 0$. Consequently, the series $z_0(i) + z_1(i) + z_2(i) + \ldots$ has all its terms zero after the $(i-a)$-th, and hence converges. Moreover, it satisfies the difference equation and the conditions at a, and accordingly is the solution of the difference equation sought when $i \geqq a$.

From (10),

$$\begin{aligned}\Delta\phi_n(i) &= \sum_{i_2=0}^{i-1}\sum_{i_3=0}^{i_2}\sum_{i_4=0}^{i_3-1}\cdots\sum_{i_{2n}=0}^{i_{2n-1}-1} p(i_2)\,p(i_4)\cdots p(i_{2n})\,(i_{2n}+1)\\ &= \sum_{i_1=0}^{i-1}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2-1}\cdots\sum_{i_{2n-1}=0}^{i_{2n-2}-1}\sum_{i_{2n}=0}^{i_{2n}-1} p(i_1)\,p(i_3)\cdots p(i_{2n-1}). \qquad (12)\end{aligned}$$

Let $2A_n = f_n(\omega) + \Delta\phi_n(\omega)$. Clearly,

$$A = 1 - A_1 + A_2 - \cdots + (-1)^{\omega} A_{\omega}. \qquad (13)$$

Consider

$$f_2(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} p(i_2)\sum_{i_3=0}^{i_2}\sum_{i_4=0}^{i_3-1} p(i_4).$$

Let $\sum\limits_{i=0}^{i-1} p(i) = P(i)$, thus defining $P(i)$; and sum by parts, considering i_2 as variable of summation. We get

$$f_2(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(P(i_1)-P(i_2))\,P(i_2).$$

Now apply similar summation by parts to $f_n(\omega)$, considering successively, as variables of summation, $i_2, i_4, \ldots, i_{2n-2}$. The above result is clearly general for any single summation, and we write

$$f_n(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1}\cdots\sum_{i_{2n}=0}^{i_{2n-1}-1}(P(i_1)-P(i_2))\cdot(P(i_2)-P(i_3))\cdots (P(i_{n-1})-P(i_n))\,P(i_n). \qquad (14)$$

Consider next

$$\Delta\phi_2(\omega) = \sum_{i_1=0}^{\omega-1} p(i_1)\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2-1} p(i_3)\sum_{i_4=0}^{i_3} 1.$$

Sum by parts, considering successively i_1 and i_3 as variables of summation.

$$\Delta\phi_2(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(P(\omega)-P(i_1))\,(P(i_1)-P(i_2)).$$

In general, letting $P(\omega) = \Omega$,

$$\Delta\phi_n(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1}\cdots\sum_{i_n=0}^{i_{n-1}-1}(\Omega-P(i_1))\,(P(i_1)-P(i_2))\cdots (P(i_{n-1})-P(i_n)). \qquad (15)$$

Combining (14) and (15),

$$2A_n = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1}\cdots\sum_{i_n=0}^{i_{n-1}-1}(\Omega-P(i_1)+P(i_n))\cdot(P(i_1)-P(i_2))\cdots (P(i_{n-1})-P(i_n)). \qquad (16)$$

Remark that when $F(i)$ is any function, $\sum\limits_{i_1=0}^{i-1}\sum\limits_{i_2=0}^{i_1} F(i_2) = \sum\limits_{i_1=0}^{i}\sum\limits_{i_2=0}^{i_1-1} F(i_2)$.

Then, from (11) and (12),

$$f_n(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1}\sum_{i_4=0}^{i_3}\cdots\sum_{i_{2n-1}=0}^{i_{2n-2}-1}\sum_{i_{2n}=0}^{i_{2n-1}} p(i_2)\,p(i_4)\cdots p(i_{2n}), \qquad (17)$$

$$\Delta\phi_n(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2}\sum_{i_4=0}^{i_3-1}\cdots\sum_{i_{2n-1}=0}^{i_{2n-2}}\sum_{i_{2n}=0}^{i_{2n-1}-1} p(i_1)\,p(i_3)\cdots p(i_{2n-1}). \qquad (18)$$

Apply summation by parts to (17) and (18), considering as variables of summation $i_1, i_3, \ldots, i_{2n-1}$ and $i_2, i_4, \ldots, i_{2n-2}$ respectively. We arrive at the following formula:

$$2A_n = \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} \sum_{i_3=0}^{i_2-1} \cdots \sum_{i_n=0}^{i_{n-1}-1} (\omega - i_1 + i_n)(i_1 - i_2) \cdots (i_{n-1} - i_n)\, p(i_1)\, p(i_2) \cdots p(i_n), \tag{19}$$

an alternative formula to (16). One draws the conclusion from it, as easily in various other ways, that if $p(i) \geqq 0$ at all points, $A_n \geqq 0$.

§ 3. *Fundamental Theorem.*

THEOREM II. *If* $p(i) \geqq 0$ *at all points, then, if* $A_n = 0$, $A_{n+1} = 0$; *if* $A_n \neq 0$, $\dfrac{A_{n+1}}{A_n} < \dfrac{n}{n+1} \dfrac{A_n}{A_{n-1}}$.

To prove the first conclusion of this theorem, we refer to (19). This formula can be written

$$A_n = \Sigma (\omega - i_1 + i_n)(i_1 - i_2) \cdots (i_{n-1} - i_n)\, p(i_1)\, p(i_2) \cdots p(i_n),$$

where Σ denotes the sum of all possible products of the form expressed, $i_1, i_2, \ldots, i_n$ taken in every possible way from $\omega - 1, \omega - 2, \ldots, 0$, subject to the restrictions $i_1 > i_2 > \cdots > i_n$. $(\omega - i_1 + i_n)(i_1 - i_2) \cdots (i_{n-1} - i_n)$ is always positive. Hence, if $A_n = 0$, each product $p(i_1) \cdots p(i_n)$ must be zero; that is, there do not exist n numbers of the set $\omega - 1, \omega - 2, \ldots, 0$ for which $p \neq 0$. But A_{n+1} is the sum of products of the form

$$(\omega - i_1 + i_{n+1})(i_1 - i_2) \cdots (i_n - i_{n+1})\, p(i_1)\, p(i_2) \cdots p(i_{n+1});$$

$i_1, i_2, \ldots, i_{n+1}$ numbers of the set $\omega-1, \omega-2, \ldots, 0$ and $i_1 > i_2 > \cdots > i_{n+1}$; and hence is zero.

For the second conclusion of the theorem we refer to (16). This can be written

$$2A_n = \Sigma (\Omega - P(i_1) + P(i_n))(P(i_1) - P(i_2)) \cdots (P(i_{n-1}) - P(i_n)),$$

where Σ denotes the sum of all products of the form expressed, the letters i_1, i_2, i_n, chosen in every possible way from the numbers $\omega-1, \omega-2, \ldots, 1, 0$, subject to the restrictions $i_1 > i_2 > i_3 > \cdots > i_n$.

If we conceive of the numbers $\omega-1, \omega-2, \ldots, 1, 0$ as equally spaced points on a circle of circumference ω, in the expression

$$(\Omega - P(i_1) + P(i_n))(P(i_1) - P(i_2)) \cdots (P(i_{n-1}) - P(i_n)),$$

the first factor is in no manner different from any other, and (16) can be written

$$2A_n = \Sigma \left[\left(\sum_{i=k_0}^{kl_1} p(i) \right) \left(\sum_{i=kl_1+}^{kl_2} p(i) \right) \cdots \left(\sum_{i=kl_{n-1}+1}^{kl_n} p(i) \right) \right],$$

where Σ denotes the sum of all possible products of the form expressed, $k_0, k_1, \ldots, k_{l_n} = k_\omega$ being the numbers $\omega-1, \omega-2, \ldots, 0$ taken always in the same cyclic order, namely $\omega-1, \omega-2, \ldots, 0$. For brevity we write

$$2A_n = \Sigma\, {}_0D_{l_1} \cdot {}_{l_1}D_{l_2} \cdots {}_{l_{n-1}}D_{l_n}.$$

Then

$$4A_n^2 = \{\Sigma\, {}_0D_{\lambda_1} \cdot {}_{\lambda_1}D_{\lambda_2} \cdots {}_{\lambda_{n-1}}D_{\lambda_n}\}\, \{\Sigma\, {}_0D_{\mu_1} \cdot {}_{\mu_1}D_{\mu_2} \cdots {}_{\mu_{n-1}}D_{\mu_n}\}, \tag{20}$$

$$4A_{n-1}A_{n+1} = \{\Sigma\, {}_0D_{\nu_1} \cdot {}_{\nu_1}D_{\nu_2} \cdots {}_{\nu_{n-2}}D_{\nu_{n-1}}\}\, \{\Sigma\, {}_0D_{\rho_1} \cdot {}_{\rho_1}D_{\rho_2} \cdots {}_{\rho_n}D_{\rho_{n+1}}\}, \tag{21}$$

where, instead of using only the letter l, we use distinct letters, λ, μ, ν, ρ.

We shall consider (20) and (21). Begin by supposing $a_1, a_2, \ldots, a_{2n}$ numbers of the succession $\omega-1, \omega-2, \ldots, 0$, and assume that among the a's there are exactly k distinct numbers, and that no number occurs more than twice among them. If $k \geqq n+1$, the product $p(a_1)p(a_2)\ldots p(a_{2n})$ will occur in the expanded right-hand members of both (20) and (21). We shall show that the ratio of its coefficient in (20) to its coefficient in (21) is greater than or equal to $\frac{n+1}{n}$.

Omitting coefficients, let $p(j_1)\ldots p(j_{n-1})$ be a term of A_{n-1} and $p(i_1)\ldots p(i_{n+1})$ a term of A_{n+1} such that $p(j_1)\ldots p(j_{n-1})\,p(i_1)\ldots p(i_{n+1})$ is identical with $p(a_1)\,p(a_2)\ldots p(a_{2n})$, and let $p(\bar{j}_1)\ldots p(\bar{j}_n)$ and $p(\bar{i}_1)\ldots p(\bar{i}_n)$ be terms of A_n such that $p(\bar{j}_1)\ldots p(\bar{j}_n)\,p(\bar{i}_1)\ldots p(\bar{i}_n)$ is identical with $p(a_1)\,p(a_2)\ldots p(a_{2n})$. We shall show that the ratio of the number of ways in which $\bar{j}_1, \ldots, \bar{j}_n, \bar{i}_1, \ldots, \bar{i}_n$ can be chosen to the number of ways in which $j_1, \ldots, j_{n-1}, i_1, \ldots, i_{n+1}$ can be chosen is greater than or equal to $\frac{n+1}{n}$. These numbers are exactly the coefficients of $p(a_1)\ldots p(a_{2n})$ in (20) and (21) respectively.

Require, first, that $p(j_1), p(j_2), \ldots, p(j_{n-1})$ be each the first term of one of the parentheses ${}_0D_{\nu_1}, {}_{\nu_1}D_{\nu_2}, \ldots, {}_{\nu_{n-2}}D_{\nu_{n-1}}$, and that $p(i_1), p(i_2), \ldots, p(i_{n+1})$ be each the first term of one of the parentheses ${}_0D_{\rho_1}, {}_{\rho_1}D_{\rho_2}, \ldots, {}_{\rho_n}D_{\rho_{n+1}}$. Under this requirement the number of ways in which $j_1, \ldots, j_n, i_1, \ldots, i_{n+1}$ can be chosen is the number of ways in which $j_1, \ldots, j_{n-1}$ can be chosen from $a_1, a_2, \ldots, a_{2n}$. The $2n-k$ numbers which occur twice among $a_1, a_2, \ldots, a_{2n}$ necessarily occur among $j_1, j_2, \ldots, j_{n-1}$. There remain $k-n-1$ of the j's which can be chosen arbitrarily from the remaining $2k-2n$ numbers. Letting $k-n = N_k$, this can be done in $\frac{2N_k(2N_k-1)\ldots(N_k+2)}{(N_k-1)!}$ ways. Similarly, the number of ways that $\bar{j}_1, \ldots, \bar{j}_n, \bar{i}_1, \ldots, \bar{i}_n$ can be chosen, requiring that $p(\bar{j}_1), \ldots, p(\bar{j}_n)$ be each the first term of one of the parentheses ${}_0D_{\lambda_1}, \ldots,$

${}_{\lambda_{n-1}}D_{\lambda_n}$, and $p(\bar{i}_1), \ldots, p(\bar{i}_n)$ be each the first term of one of the parentheses ${}_0D_{\mu_1}, \ldots, {}_{\mu_{n-1}}D_{\mu_n}$, is $\dfrac{2N_k(2N_k-1)\ldots(N_k+1)}{N_k!}$. The second is larger in the ratio $\dfrac{N_k+1}{N_k}$. But $N_k \leqq n$, and hence $\dfrac{N_k+1}{N_k} \geqq \dfrac{n+1}{n}$.

We generalize as follows: Instead of requiring that each p be the first term of a parenthesis, let us require that $p(a_1)$ be the λ-th, $p(a_2)$ the μ-th, $\ldots$, $p(a_{2n})$ the ζ-th. For convenience we shall refer to $j_1, \ldots, j_{n-1}$ and $\bar{j}_1, \ldots, \bar{j}_n$ as the sets J and to $i_1, \ldots, i_{n+1}$ and $\bar{i}_1, \ldots, \bar{i}_n$ as the sets I. As above, those a's occurring twice among $a_1, \ldots, a_{2n}$ necessarily occur in both the sets J and I. Consider them as fixed. We proceed as before, choosing the remainder of the sets J.

It may happen that the fact that a_m lies in the sets J (or I) requires that $a_{m+\nu}$ lie in the corresponding sets I (or J). Thus, suppose that a_m is the ρ-th term of a parenthesis and $a_{m+\nu}$ the $\bar{\rho}$-th, and suppose that $\bar{\rho} > \nu$; then, if a_m is one of the set J, in order for $p(a_{m+\nu})$ to lie in a different parenthesis from $p(a_m)$, as it must, it must necessarily be a member of the set I. Moreover, the fact that a_m lies in the set J can require that only one of the a's lie in the sets I; for, suppose that $p(a_{m+\mu})$ is the $\bar{\bar{\rho}}$-th term of a parenthesis and $\bar{\bar{\rho}} > \mu \geqq \nu$, then $p(a_{m+\nu})$ and $p(a_{m+\mu})$ belong to the sets I and lie in different parentheses. Hence, $\bar{\bar{\rho}} \leqq \mu - \nu$ but $\bar{\bar{\rho}} > \mu$, a contradiction.

In the way that we are choosing the sets J, let us suppose all $p(a)$'s that impose any restriction on others as fixed. Let this number be L. Then there are thereby fixed R in the sets I, and necessarily $R \leqq L$. The remaining a's can now be distributed in sets J and I at pleasure. This can be done in (21) and (20) in

$$\frac{(2N_k - L - R)(2N_k - L - R - 1)\ldots(N_k - R + 2)}{(N_k - L - 1)!}$$

and

$$\frac{(2N_k - L - R)(2N_k - L - R - 1)\ldots(N_k - R + 1)}{(N_k - L)!}$$

ways respectively. The second is the larger in the ratio $\dfrac{N_k - R + 1}{N_k - L}$, which is greater than $\dfrac{n+1}{n}$. We thus conclude that the coefficient of $p(a_1)\ldots p(a_{2n})$ in (20) is greater than its coefficient in (21) by a ratio greater than or equal to $\dfrac{n+1}{n}$.

We have considered $k \geqq n+1$, which exhausts the terms of (21). There are in addition in (20) terms of the form $p(a_1)\ldots p(a_{2n})$, where $k = n$; that

7

is, terms of the form $(p(a_1) \ldots p(a_n))^2$. These are not all zero as $A_n \neq 0$. All coefficients are positive, and hence we conclude

$$A_n^2 > \frac{n+1}{n} A_{n+1} A_{n-1},$$

from which we immediately draw the desired conclusion

$$\frac{A_{n+1}}{A_n} < \frac{n}{n+1} \frac{A_n}{A_{n-1}}. \tag{22}$$

From theorem II one readily proves the following:

THEOREM III. *If $p(i) \geqq 0$ at all points,*
when $-1 + A_2 - A_3 + \ldots + A_{2n} \leqq 0, \quad A < 1;$
when $2 - A_1 + A_2 - \ldots - A_{2n-1} \geqq 0, \quad A > -1;$
when $2 - A_1 + A_2 - \ldots + A_{2n} \leqq 0, \quad A < -1;$
when $-A_1 + A_2 - A_3 + \ldots - A_{2n-1} \geqq 0, \; A > 1.$

§ 4. *The Calculation of A_2 and A_3.*

The problem now proposed is the calculation of A_2, A_3, etc., with as little labor as possible. We retain the supposition $p \geqq 0$ at all points.

From (16),

$$A_2 = \tfrac{1}{2} \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} (\Omega - P(i_1) + P(i_2))(P(i_1) - P(i_2)) = \tfrac{1}{2}\Omega \sum_{i=0}^{\omega-1} iP(i) - \tfrac{1}{2} \sum_{i=0}^{\omega-1} i(P(i))^2$$
$$+ \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} P(i_1)P(i_2) - \tfrac{1}{2}\Omega \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} P(i_2) - \tfrac{1}{2} \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} (P(i_2))^2.$$

This expression can be still farther reduced by summing by parts those terms in which two Σ's occur. We obtain

$$A_2 = \Omega \sum_{i=0}^{\omega-1} iP(i) - \frac{\omega}{2} \sum_{i=0}^{i-1} (P(i))^2 + \tfrac{1}{2} \Big(\sum_{i=0}^{\omega-1} P(i) \Big)^2 - \Omega \frac{\omega - 1}{2} \sum_{i=0}^{\omega-1} P(i).$$

From this we easily verify the formula

$$A_2 = \Omega^2 \frac{\omega^2 - 1}{24} - \tfrac{1}{2} R, \tag{23}$$

where

$$R = \omega \sum_{i=0}^{\omega-1} \Big(P(i) - \Omega \frac{i}{\omega} \Big)^2 - \Big[\sum_{i-0}^{\omega-1} \Big(P(i) - \Omega \frac{i}{\omega} \Big) \Big]^2. \tag{24}$$

If $a(i)$ is any real function,

$$\omega \sum_{i=0}^{\omega-1} (a(i))^2 \geqq \Big(\sum_{i=0}^{\omega-1} a(i) \Big)^2.$$

Hence $R(i) \geqq 0$. Consequently we can use $\dfrac{\omega^2 - 1}{24} \Omega^2$ as a superior limit for A_2, or more loosely $\frac{\omega^2\Omega^2}{24}$. But from (16), $A_1 = \frac{\omega\Omega}{2}$; hence

$$A_2 < \tfrac{1}{6} A_1^2.$$

With this last result we can proceed as is done by Liapounoff (§ 14), obtaining the same result as is obtained by him.

We have defined $P(i)$ as $\sum_{i=0}^{i-1} p(i)$, but we might equally well have defined $P(i)$ by the equation $P(i) = \Sigma p(i)$, where Σ denotes the indefinite sum, retaining the notation $\Omega = \sum_{i=0}^{\omega-1} p(i)$, since in the formula for A_n, (16), $P(i)$ occurs only in the combination $(P(i_j) - P(i_k))$. Let us particularize $P(i)$ by choosing the arbitrary constant of summation so that

$$\sum_{i=0}^{\omega-1} P(i) = \Omega \frac{\omega - 1}{2}. \tag{25}$$

It is immediate that

$$\sum_{i=0}^{\omega-1} (P(i) - \Omega i/\omega) = 0, \tag{26}$$

and we get as a simplified formula

$$R = \omega \sum_{i=0}^{\omega-1} (P(i) - \Omega i/\omega)^2. \tag{27}$$

$(P(i) - \Omega i/\omega)$ has the period ω, since increasing i by ω increases both terms of the expression by Ω. The same thing is true of its sum; that is,

$$\sum_{i=0}^{i-1} (P(i) - \Omega i/\omega)$$

has the period ω. For, if we increase i by ω, we add

$$\sum_{i=i}^{i+\omega-1} (P(i) - \Omega i/\omega) = \sum_{i=0}^{\omega-1} (P(i) - \Omega i/\omega) = 0,$$

by (26).

Now let $P(i) - \Omega i/\omega = \Omega \Delta\theta(i)$, where $\Delta\theta(i)$ denotes the first difference of a function θ. We have just shown that θ has the period ω. From (23) and (27),

$$A_2 = \Omega^2 \left\{ \frac{\omega^2 - 1}{24} - \tfrac{1}{2}\omega \sum_{i=0}^{\omega-1} (\Delta\theta(i))^2 \right\}. \tag{28}$$

For brevity let $i/\omega + \Delta\theta(i) = Q(i)$; then (16) gives

$$A_n = a_n \Omega^n,$$

where

$$a_n = \tfrac{1}{2} \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} \cdots \sum_{i_n=0}^{i_{n-1}-1} (1 - Q(i_1) + Q(i_n))(Q(i_1) - Q(i_2)) \cdots (Q(i_{n-1}) - Q(i_n)).$$

Hence,

$$a_3 = \tfrac{1}{2} \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} \sum_{i_3=0}^{i_2-1} (1 - Q(i_1) + Q(i_3))(Q(i_1) - Q(i_2))(Q(i_2) - Q(i_3)). \tag{29}$$

It is possible to greatly simplify this expression. The summand reduces to

$$Q(i_1)\,Q(i_2)+Q(i_2)\,Q(i_3)-Q(i_1)\,Q(i_3)-(Q(i_2))^2+(Q(i_3)-Q(i_2))\,(Q(i_1))^2$$
$$+(Q(i_1)-Q(i_3))\,(Q(i_2))^2+(Q(i_2)-Q(i_1))\,(Q(i_3))^2.$$

Distribute the sign of summation and apply to each term summation by parts or perform obvious summation. We obtain the following results:

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} Q(i_1)\,Q(i_2)=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} i_2\,Q(i_1)\,Q(i_2),$$
$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} Q(i_2)\,Q(i_3)=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} (\omega-i_1-1)\,Q(i_1)\,Q(i_2),$$
$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} Q(i_1)\,Q(i_3)=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} (i_1-i_2-1)\,Q(i_1)\,Q(i_2),$$
$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} (Q(i_2))^2=\sum_{i=0}^{\omega-1} (\omega-i-1)\,i\,(Q(i))^2,$$
$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} (Q(i_1))^2\,Q(i_3)=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} (i_1-i_2-1)\,(Q(i_1))^2\,Q(i_2),$$
$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} (Q(i_1))^2\,Q(i_2)=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} i_2\,(Q(i_1))^2\,Q(i_2),$$
$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} Q(i_1)\,(Q(i_2))^2=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} Q(i_1)\,(Q(i_2))^2\,i_2,$$
$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} (Q(i_2))^2\,Q(i_3)=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} (\omega-i_1-1)\,(Q(i_1))^2\,Q(i_2),$$
$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} Q(i_2)\,Q(i_3)^2=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} (\omega-i_1-1)\,Q(i_1)\,(Q(i_2))^2,$$
$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} Q(i_1)\,(Q(i_3))^2=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} (i_1-i_2-1)\,Q(i_1)\,(Q(i_2))^2.$$

Collecting,

$$a_3=\tfrac{1}{2}\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}[Q(i_1)\,Q(i_2)\,(\omega+2\,i_2-2\,i_1)+(Q(i_1))^2\,Q(i_2)\,(2\,i_1-2\,i_2-\omega)$$
$$+Q(i_1)\,(Q(i_2))^2\,(\omega+2\,i_2-2\,i_1)]-\tfrac{1}{2}\sum_{i=0}^{\omega-1}(\omega-i-1)\,i\,(Q(i))^2. \tag{30}$$

By means of the formula for summation by parts, one proves easily:

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} Q(i_1)Q(i_2)=\tfrac{1}{2}\Big(\sum_{i=0}^{\omega-1}Q(i)\Big)^2-\tfrac{1}{2}\sum_{i=0}^{\omega-1}(Q(i))^2,$$
$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(Q(i_1))^2\,Q(i_2)\,i_2=\Big(\sum_{i=0}^{\omega-1}i\,Q(i)\Big)\Big(\sum_{i=0}^{\omega-1}(Q(i))^2\Big)-\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1}i_1\,Q(i_1)\,(Q(i_2))^2$$
$$=\Big(\sum_{i=0}^{\omega-1}i\,Q(i)\Big)\Big(\sum_{i=0}^{\omega-1}(Q(i))^2\Big)$$
$$-\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}i_1\,Q(i_1)\,(Q(i_2))^2-\sum_{i=0}^{\omega-1}i\,(Q(i))^3.$$

Similarly,

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} Q(i_1)\,(Q(i_2))^2\, i_2 = \Big(\sum_{i=0}^{\omega-1} i\,(Q(i))^2\Big)\Big(\sum_{i=0}^{\omega-1} Q(i)\Big) - \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} i_1\,(Q(i_1))^2\, Q(i_2) - \sum_{i=0}^{\omega-1} i\,(Q(i))^3.$$

Substituting these values in (30),

$$\begin{aligned} a_3 = {} & \frac{\omega}{4}\Big(\sum_{i=0}^{\omega-1} Q(i)\Big)^2 + \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(i_2-i_1)\,Q(i_1)\,Q(i_2) \\ & + \Big(\sum_{i=0}^{\omega-1} i\,(Q(i))^2\Big)\Big(\sum_{i=0}^{\omega-1} Q(i)\Big) - \Big(\sum_{i=0}^{\omega-1} i\,Q(i)\Big)\Big(\sum_{i=0}^{\omega-1}(Q(i))^2\Big) \\ & + \frac{\omega}{2}\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} Q(i_1)\,Q(i_2)\,(Q(i_2)-Q(i_1)) - \frac{\omega-1}{2}\sum_{i=0}^{\omega-1} i\,(Q(i))^2 \\ & - \frac{\omega}{4}\sum_{i=0}^{\omega-1}(Q(i))^2 + \tfrac{1}{2}\sum_{i=0}^{\omega-1} i^2\,(Q(i))^2. \end{aligned} \tag{31}$$

Moreover, $Q(i) = \frac{i}{\omega} + \Delta\theta(i) = \frac{i}{\omega} + \Big(P(i) - \Omega\frac{i}{\omega}\Big)$. Hence, by (26),

$$\sum_{i=0}^{\omega-1} Q(i) = \sum_{i=0}^{\omega-1}\frac{i}{\omega} = \frac{\omega-1}{2}.$$

Substituting in (31),

$$\begin{aligned} a_3 = {} & \frac{(\omega-1)^2\omega}{16} + \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(i_2-i_1)\,Q(i_1)\,Q(i_2) \\ & + \tfrac{1}{2}\sum_{i=0}^{\omega-1} i^2\,(Q(i))^2 - \Big(\sum_{i=0}^{\omega-1} i\,Q(i)\Big)\Big(\sum_{i=0}^{\omega-1}(Q(i))^2\Big) \\ & + \frac{\omega}{2}\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} Q(i_1)\,Q(i_2)\,(Q(i_2)-Q(i_1)) - \frac{\omega}{4}\sum_{i=0}^{\omega-1}(Q(i))^2. \end{aligned} \tag{32}$$

We have only defined the function θ by its first difference. We can particularize by letting $\theta(0) = \theta(\omega) = 0$. Under this assumption, the above expression for a_3 permits of great simplification. We treat the different sums occurring separately, applying summation by parts, collecting and simplifying, arriving at the following result:

$$\begin{aligned} a_3 = {} & \frac{1}{\omega}\Big[\frac{\omega^4}{720} - \frac{\omega^2}{144} + \frac{1}{180}\Big] + 2\sum_{i=0}^{\omega-1}(\theta(i))^2 - \frac{\omega^2+2}{12}\sum_{i=0}^{\omega-1}(\Delta\theta(i))^2 \\ & - \frac{\omega}{2}\sum_{i=0}^{\omega-1}\theta(i)\,(\Delta\theta(i))^2 - \frac{\omega}{2}\sum_{i=0}^{\omega-1}\theta(i+1)\cdot(\Delta\theta(i))^2. \end{aligned} \tag{33}$$

We know that

$$A_3 = \Omega^3\, a_3. \tag{34}$$

To obtain the actual formulas desired, we proceed thus:

$$\Omega\,\Delta\theta(i) = P(i) - \Omega\frac{i}{\omega} = \sum_{i=0}^{i-1} p(i) - \Omega\frac{i}{\omega} + C,$$

where C is determined so that $\sum_{i=0}^{\omega-1} P(i) = \Omega \frac{\omega-1}{2}$.

$$\Omega \Delta^2 \theta(i) = p(i+1) - \frac{\Omega}{\omega}.$$

Let $\Omega \Delta^2 \theta(i) = \Delta^2 \phi(i+1)$, where $\Delta^2 \phi(i)$ denotes the second difference of a function $\phi(i)$. Let $\frac{\Omega}{\omega} = c$. Then $\Delta^2 \phi(i+1) = c\omega \Delta^2 \theta(i)$. Determine the arbitrary constants of summation so that $\Delta \phi(i+1) = c\omega \Delta \theta(i)$ and $\phi(i+1) = c\omega\theta(i)$. Like $\Delta\theta(i)$ and $\Delta^2\theta(i)$, $\Delta\phi(i)$ and $\Delta^2\phi(i)$ have the period ω. Bear in mind that, if $f(i)$ has the period ω, $\sum_{i=k}^{i=k+\omega-1} f(i)$ is independent of k, and substitute in (16) for A_1 and in (28) and (34) for A_2 and A_3 respectively, using the value of a_3 given by (33):

$$A_1 = \frac{c\omega^2}{2}, \tag{35}$$

$$A_2 = c^2\omega^2\left[\frac{\omega^2-1}{24}\right] - \tfrac{1}{2}\omega \sum_{i=0}^{\omega-1} (\Delta\phi(i))^2, \tag{36}$$

$$\begin{aligned} A_3 = c^3\omega^2\left[\frac{\omega^4}{720} - \frac{\omega^2}{144} + \frac{1}{180}\right] + 2c\omega \sum_{i=0}^{\omega-1} (\phi(i))^2 - c\frac{\omega^3+2\omega}{12} \sum_{i=0}^{\omega-1} (\Delta\phi(i))^2 \\ - \frac{\omega}{2}\sum_{i=0}^{\omega-1} \phi(i)(\Delta\phi(i))^2 - \frac{\omega}{2}\sum_{i=0}^{\omega-1} (\Delta\phi(i))^2\phi(i+1). \end{aligned} \tag{37}$$

We proceed by determining $c = \frac{\Omega}{\omega}$, then $\Delta^2\phi(i) = p(i) - ci$, then $\Delta\phi(i) = \Sigma\Delta^2\phi(i)$, where that particular sum is chosen which will cause $\phi(i)$ to be periodic.

$\phi(i) = \Sigma\Delta\phi(i)$, determined so that $\phi(1) = 0$.

Formulas (35), (36) and (37) are easily applicable when p is expressed as a trigonometric sum, which development is always, theoretically at least, possible.